Multiple choice questions

1) (C), $x^3 - 3^3 = (x-3)(x^2+3x+9)$

2) (A), $x = \frac{3\times 8 + 2\times -2}{3+2} = \frac{20}{5} = 4$

3) (B), $x^3 - \sum \alpha . x^2 + \sum \alpha\beta . x - \prod \alpha$

4) (C), since D: $-2 \le x \le 2$, R: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$.

5) (D), $6! \times 2$.

6) (A), $n = 4, \therefore T = \frac{2\pi}{n} = \frac{\pi}{2}, A^2 = 9, \therefore A = 3.$

7) (C), $\int \sin^2 3x dx = \frac{1}{2}\int (1-\cos 6x) dx.$

8) (D), $P(x) = (x+1)(x-3)Q(x) + 2x + 7, P(3) = 13$

9) (D), $\frac{d}{dx}\left(\cos^{-1} ax\right) = \frac{-a}{\sqrt{1-(ax)^2}}$

10) (B), $\angle ABT = 90° - \frac{\theta}{2}$ (angle sum in isos. Δ)

and $\angle APB = \angle ABT$ (alternate segment angles)

Question 11

(a) $\int_0^3 \frac{1}{9+x^2} dx = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3 = \frac{1}{3}\times\frac{\pi}{4} = \frac{\pi}{12}.$

(b) $\frac{d}{dx}\left(x^2 \tan x\right) = 2x\tan x + x^2 \sec^2 x.$

(c) $\frac{x}{x-3} < 2$

$x(x-3) < 2(x-3)^2$

$(x-3)\left(x - 2x + 6\right) < 0$

$(x-3)(-x+6) < 0$

$x < 3$ or $x > 6$.

(d) $u = 2 - x, du = -dx.$

When $x = 1, u = 1$; when $x = 2, u = 0$.

$I = \int_1^0 (2-u)u^5(-du) = \int_0^1 \left(2u^5 - u^6\right) du$

$= \left[\frac{u^6}{3} - \frac{u^7}{7}\right]_0^1 = \frac{1}{3} - \frac{1}{7} = \frac{4}{21}.$

(e) ${}^8C_3 \times {}^{10}C_4 = 11760.$

(f) (i) $\left(2x^3 - \frac{1}{x}\right)^{12} = \frac{(2x^4-1)^{12}}{x^{12}}.$

The constant term is $\frac{{}^{12}C_3 (2x^4)^3 (-1)^9}{x^{12}}$

$= -{}^{12}C_3 (2)^3 = -1760.$

(ii) $\left(2x^3 - \frac{1}{x}\right)^n = \frac{(2x^4-1)^n}{x^n}.$

$T_{k+1} = \frac{{}^nC_k (2x^4)^k (-1)^{n-k}}{x^n}.$

$\therefore n = 4k$, k is a non-negative integer.

Question 12

(a) Let $n=1, 2^3-3=8-3=5$, hence, divisible by 5.

Assume $2^{3n}-3^n=5M$, where is an integer.

$\therefore 2^{3n}=5M+3^n$.

Required to prove that $2^{3(n+1)}-3^{n+1}$ is divisible by 5.

$$\begin{aligned}2^{3(n+1)}-3^{n+1} &= 2^3\times 2^{3n}-3\times 3^n\\ &= 8(5M+3^n)-3\times 3^n\\ &= 40M+5\times 3^n\\ &= 5\left(8M+3^n\right),\text{ which is divisible by 5.}\end{aligned}$$

$\therefore$ By the principle of Induction, $2^{3n}-3^n$ is divisible by 5 for all $n\geq 1$.

(b) (i) $4x-3\geq 0, \therefore x\geq \frac{3}{4}$.

(ii) $f: y=\sqrt{4x-3}$.

$f^{-1}: x=\sqrt{4y-3}$.

$x^2=4y-3$.

$\therefore y=\frac{x^2+3}{4}, x\geq 0$.

(iii) $\frac{x^2+3}{4}=x$

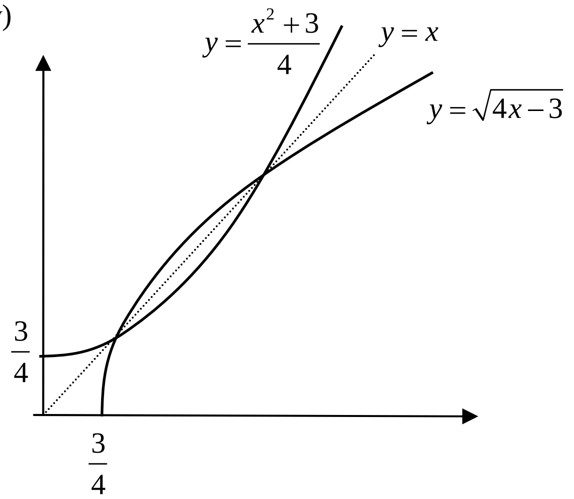
$x^2-4x+3=0$

$(x-1)(x-3)=0$

$\therefore x=1,3$

$\therefore$ Points of intersection (1,1) and (3,3).

(iv)



(c) (i) $\Pr(W)=\Pr(L), \Pr(D)=\frac{1}{5}$.

$\Pr(W)+\Pr(L)+\Pr(D)=1$.

$\therefore \Pr(W)=\frac{2}{5}$.

(ii) ${}^6C_3\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^3=0.2765$.

(d) (i) $AC\perp BC, \therefore \frac{y}{-t}\times\frac{k}{t}=-1, \therefore y=\frac{t^2}{k}$.

$x_P=x_C=t, y_P=y_B=y=\frac{t^2}{k}$.

(ii) The equation of the parabola is $y=\frac{x^2}{k}$, i.e. $ky=x^2$.

$4a=k, \therefore a=\frac{k}{4}$.

$\therefore$ Focus $\left(0,\frac{k}{4}\right)$.

Question 13

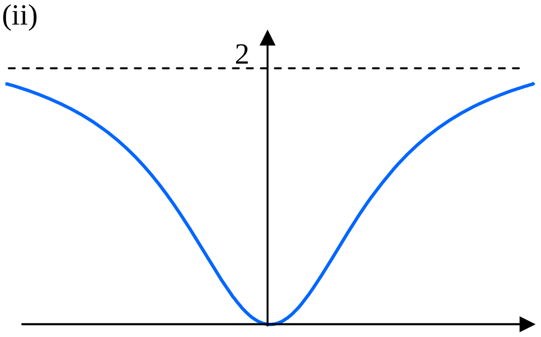
(a) $\sin\left(2\cos^{-1}\frac{2}{3}\right)=\sin 2\alpha = 2\sin\alpha\cos\alpha$

$=2\times\frac{\sqrt{5}}{3}\times\frac{2}{3}=\frac{4\sqrt{5}}{9}.$

(b) (i) $y=\frac{2x^2}{x^2+9}=\frac{2}{1+\frac{9}{x^2}}\to 2$ as $x\to\infty.$

$\therefore$ Horizontal asymptote $y=2$.

(ii)



(c) (i) $x=5+6\cos 2t+8\sin 2t$

$=5+10\cos\left(2t-\tan^{-1}\frac{4}{3}\right).$

$\dot{x}=-20\sin\left(2t-\tan^{-1}\frac{4}{3}\right).$

$\ddot{x}=-40\cos\left(2t-\tan^{-1}\frac{4}{3}\right)=-4(x-5),\therefore$ SHM.

(ii) $0=5+10\cos\left(2t-\tan^{-1}\frac{4}{3}\right).$

$\cos\left(2t-\tan^{-1}\frac{4}{3}\right)=-\frac{1}{2}.$

$2t-\tan^{-1}\frac{4}{3}=\frac{2\pi}{3}.$

$\therefore t=\frac{1}{2}\tan^{-1}\frac{4}{3}+\frac{\pi}{3}$ sec $\simeq 1.5$ sec.

(d) (i) $\frac{dC}{dt}=1.4\left(e^{-0.2t}-0.2te^{-0.2t}\right)=1.4e^{-0.2t}\left(1-0.2t\right)$

$\frac{dC}{dt}=0$ gives $t=\frac{1}{0.2}=5$ hours.

$\frac{d^2C}{dt^2}=1.4\left(-0.2e^{-0.2t}(1-0.2t)-0.2e^{-0.2t}\right)$

$=-0.28e^{-0.2t}\left(2-0.2t\right)<0$ when $t=5$.

$\therefore$ The maximum occurs when $t=5$ hours.

(ii) $C_2=20+\frac{0.3-1.4\times 20\times e^{-4}}{1.4e^{-4}(1-4)}=22.8$ hrs.

Note: Newton's method is based on the equation of the tangent: $y-y_1=f'(x_1)\left(x-x_1\right)$.

Usually we find the approximation when $y=0$.

Here, $y=0.3$

$\therefore x-x_1=\frac{y-y_1}{f'(x_1)}$

$x=x_1+\frac{y-y_1}{f'(x_1)}.$

Question 14

(a) (i) $CS \perp AD$ (semi-circle angle)

For the same reason, $CT \perp BT$ and $AD \perp DB$

$\therefore CTDS$ is a rectangle (three right angles)

(ii) $MS = MC$ (= radii)

$XS = XC$ (in a rectangle, the diagonals bisect)

MX is common

$\therefore \Delta MXS \equiv \Delta MXC$ (SSS)

(iii) $\angle MSX = \angle MCX$ (corresponding angles in congruent triangles)

$\therefore \angle MSX = 90°$

$\therefore ST$ is a tangent (tangent is perpendicular to the radius at the point of contact).

(b) (i) At maximum height, $\dot{y} = 0$

$70\sin\theta - 9.8t = 0$

$t = \dfrac{70\sin\theta}{9.8}$ s.

$y = 70t\sin\theta - 4.9t^2$

$= \dfrac{4900\sin^2\theta}{9.8} - \dfrac{4.9 \times 4900\sin^2\theta}{9.8^2}$

$= 250\sin^2\theta.$

(ii) Substitute $t = \dfrac{70\sin\theta}{9.8}$ into $x = 70t\cos\theta$

$x = \dfrac{4900\sin\theta\cos\theta}{9.8} = 500\sin\theta\cos\theta$

$= 250\sin 2\theta.$

(iii) Solving $250\sin^2\theta \geq 150$ gives $\sin^2\theta \geq \dfrac{3}{5}$

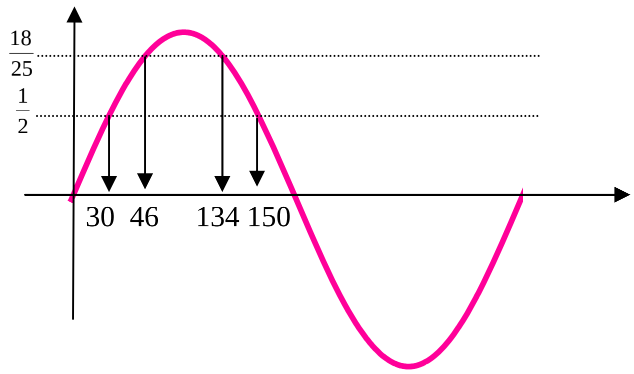
$\sin\theta \geq \dfrac{\sqrt{15}}{5}, \therefore 50.8° \leq \theta \leq 129.2°.$ (1)

Solving $125 \leq 250\sin 2\theta \leq 180$ gives

$\dfrac{1}{2} \leq \sin 2\theta \leq \dfrac{18}{25}$

$30° + k.360° \leq 2\theta \leq 46° + k.360°$

or $134° + k.360° \leq 2\theta \leq 150° + k.360°.$



Dividing by 2,

$15° + k180° \leq \theta \leq 23° + k180°$

or $67° + k180° \leq \theta \leq 75° + k180°.$

$\therefore$ To satisfy (1), $67° \leq \theta \leq 75°.$

(c) (i) $\cos\alpha = \dfrac{BG}{u}, \therefore BG = u\cos\alpha.$

$\sin\alpha = \dfrac{PG}{u}, \therefore PG = u\sin\alpha.$

$AG^2 = BG^2 + 1^2 - 2BG\cos 60°$

$= BG^2 + 1 - BG = u^2\cos^2\alpha + 1 - u\cos\alpha.$

$r^2 = PG^2 + AG^2$

$= u^2\sin^2\alpha + u^2\cos^2\alpha + 1 - u\cos\alpha$

$= u^2(\sin^2\alpha + \cos^2\alpha) + 1 - u\cos\alpha$

$= u^2 + 1 - u\cos\alpha.$

$\therefore r = \sqrt{1 + u^2 - u\cos\alpha}.$

(ii) $\dfrac{dr}{dt} = \dfrac{dr}{du}\dfrac{du}{dt} = \dfrac{2u - \cos\alpha}{2\sqrt{1 + u^2 - u\cos\alpha}}\dfrac{du}{dt}$

When $t = 5$ min., $u = 30$ km, $\dfrac{du}{dt} = 6$ km/min.,

$\dfrac{dr}{dt} = \dfrac{60 - \cos\alpha}{2\sqrt{901 - 30\cos\alpha}} \times 6$

$= \dfrac{180 - 3\cos\alpha}{\sqrt{901 - 30\cos\alpha}}$ km/min.